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Calculation of Multidisturber Crosstalk Probabilities—Application to Subscriber-Loop Gain

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The possibility that excessive crosstalk limits the application of gain to long voice-frequency loops was studied. This problem was examined by means of a probability model incorporating data on such random variables as activity coincidence of disturbing and disturbed subscribers, disturber volume level, coupling path loss, listener acuity, and various noise sources.

A new approach was used with regard to the probability distributions of random variables: the probability distributions were modelled in detail for the calculation of the necessary functions, sums, and functions of sums of random variables. Results showed that gain of 6 dB or less is acceptable, and 9 dB is unjustifiably excessive. The approach used provides the most accurate calculation possible with available data, and is anticipated to be convenient not only for similar crosstalk evaluations but also for other nongaussian probability problems.

I. INTRODUCTION

An important concern in speech transmission is the avoidance of crosstalk. When it is intelligible, it is a potential violation of the telephone subscriber's privacy; when not intelligible, it is nevertheless an annoyance, especially if syllabic in nature. In subscriber loop equipment, as a result of the permanent assignment of a pair to each customer, such crosstalk situations might tend to be dedicated to a specific disturbing talker and a specific disturbed listener, who might even be known to each other. Therefore, the random occurrences of intelligible crosstalk must be limited to a probability that is very small indeed.

On long voice-frequency subscriber loops, it is often desirable to overcome excessive attenuation of the voice signal by applying some gain at the central office. The higher signal level, however, increases the probability that intelligible crosstalk will be heard. The existing gain limit of 6 dB was suspected to be already the maximum possible without risking excessive crosstalk probabilities. Therefore, desired increases to as much as 9 dB were considered problematic, and this analysis was undertaken to evaluate it.

Intelligible crosstalk results from a fortuitous combination of several random variables, such as the activity coincidence of disturbing and disturbed subscribers, disturber volume level, coupling path loss, listener acuity, and various noise sources. The ill-defined, stochastic nature of the problem has always necessitated various approximations. For the present analysis, an approach more exact than previously used was needed for two reasons. First, preliminary calculations with these conservative approximations yielded unfavorable results even for low values of gain. Second, the probabilities involved were extremely small. Therefore, if there was any way to justify more than 6-dB gain, the least conservative, most exact, calculation would be the most likely to do so.

Most of the complications in crosstalk problems arise from the difficulty of measuring and analyzing the probability distributions of the random variables; these make the problem in its entirety seem formidable indeed. Fortunately, because of its importance, past work has shed much light on the construction of a reasonable analytical framework and on the estimation of the distributions. Aspects of this study, especially those described in Sections 2.1 and 2.2, continue in the line of analysis used most recently by D. H. Morgen,[1] T. C. Spang, B. E. Davis, and M. G. Mugglin of Bell Laboratories.

The main new development in this paper is the calculation of crosstalk in terms of nongaussian probability distributions, where appropriate for improved accuracy. The data is therefore better represented than if approximating gaussian distributions had been fitted to it, a point especially significant when evaluating the extreme "tails" of distributions, as required in this case. The improvements arising from this approach include:

(*i*) The exact distribution of measured coupling losses can be considered, rather than an estimation based on worst-case coupling or other approximations.

(*ii*) The threshold for intelligibility can be represented as dependent on an exactly calculated "power sum" of noise sources.

(*iii*) Distributions can be truncated when appropriate, as for example at the 30-dBrnC noise limit. This avoids taking advantage of one plant deficiency, excessive noise, to mask another, excessive crosstalk.

The problem is addressed in terms of the general crosstalk probability model summarized in Section II. Section III explains in more

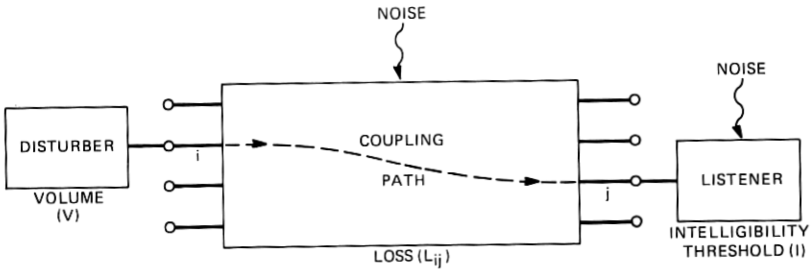

Fig. 1—General crosstalk coupling model.

detail the calculation of the total noise distribution from the distributions of the individual noise sources, after which the intelligibility threshold can be determined. The data for the distributions of random variables is discussed in Section IV, and Section V deals with some of the computational aspects involved. Finally, the results are discussed in Section VI.

II. GENERAL CROSSTALK PROBABILITY MODEL

2.1 The possibility of crosstalk

The essentials of the crosstalk problem can be described very simply and are depicted in Fig. 1. There is a disturber of volume V active on channel i, a disturbed subscriber is listening on channel j, and a coupling path with loss L^{ij} connects them. The listening acuity of the disturbed subscriber is characterized by his intelligibility threshold I.

Usually there are several possible coupling paths. Figure 2 shows the far-end and near-end crosstalk paths involved when voice-frequency gain is used on subscriber loops. However, it is easy to show that the far-end crosstalk probability is small (see Appendix A). The type of crosstalk path considered most likely to be disturbing with voice-

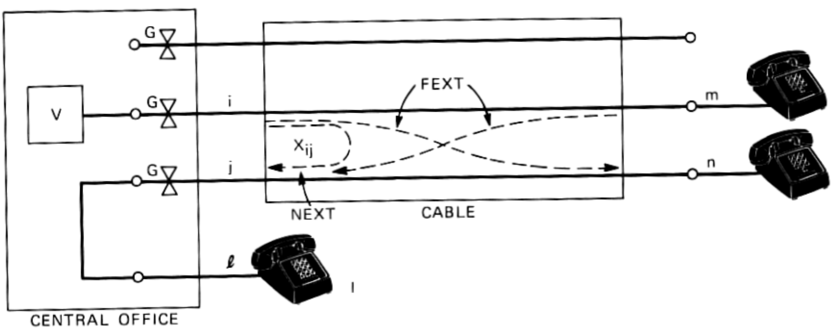

Fig. 2—VFG cable crosstalk coupling path.

frequency gain was near-end crosstalk (NEXT) at the central office (CO) because the gain appears twice in the crosstalk path. Thus, L^{ij} in Fig. 2 is seen to include the coupling loss X^{ij} in the cable, reduced by twice the gain. When all quantities are expressed in dB,

$$L^{ij} = X^{ij} - 2G.$$

The disturbed listener is considered to be at zero loss from the CO, so that crosstalk is most likely to be heard.

Crosstalk by way of the coupling path L^{ij} from disturbing pair $i \neq j$ to listener l is possible when both are active and subscriber l is using pair j. The probability of coincident activity of two independent channels has been usually given as

$$P_A^{il} = 1 - \left[\frac{\lambda}{\mu + \lambda}\right]^2, \tag{1}$$

where

μ = mean holding time of call,
λ = mean quiet interval between calls,
A denotes coincident activity.

This was a result of modeling holding times and quiet intervals as exponentially distributed; a derivation under more basic assumptions is presented in Appendix B. From (1), the probability of coincident activity for rural offices has been evaluated, assuming traffic to be somewhat heavier than average but uniformly distributed over all channels,

$$P_A^{il} = P_A = 0.167.$$

With P^{lj} denoting the probability that the disturbed subscriber l is using channel j, the probability that crosstalk can reach him by way of path L^{ij}, that is, the probability of coincident activity over the path, becomes

$$P_A^{ij} = P_A^{il} P^{lj}, \qquad i \neq j.$$

In this case, the subscribers on all cable pairs j were considered equally likely to be called by subscriber l, which leads to

$$P^{lj} = \frac{1}{N}$$

$$P_A^{ij} = P_A \cdot \frac{1}{N} \cdot \tag{2}$$

It may be noted that both the talker volume and the noise distributions have been implicitly assumed to be identical and independent for all disturbers and paths, for which reason neither V nor N has been carrying the superscripts i and j. There seems little reason to assume otherwise, although, if desired, V and N could be treated in a multidisturber context similar to that of L.

2.2 The intelligibility of crosstalk

The condition under which crosstalk over the path L^{ij} becomes intelligible to the disturbed subscriber is, in dB,

$$V - L^{ij} > I. \tag{3}$$

This relation can be elaborated to include the dependence of the intelligibility threshold I on the masking effect of all noise sources:[1]

(i) Circuit noise N_C
(ii) Room noise N_R
(iii) A fit parameter N_{TH} (which can be somewhat loosely interpreted as residual noise in the auditory system, defining the hearing threshold).

The total noise level is the sum of individual noise sources when expressed in units of power, rather than dB:

$$10^{N/10} = 10^{N_1/10} + 10^{N_2/10} + \cdots + 10^{N_K/10} = \sum_k 10^{N_k/10} \tag{4a}$$

or, in short,

$$N = N_1 +_p N_2 +_p \cdots +_p N_k, \tag{4b}$$

where the operator $+_p$ in (4b) is defined by (4a). In general, the noise sources must be specified not deterministically but rather by their probability distributions. The resulting distribution of the total noise level N can be calculated as described in Section III; we may assume for now that N is known. Then the intelligibility threshold I can be modeled as the following linear function of N (see Section 4.3),

$$I = I_0 + N. \tag{5}$$

Thus, the condition for crosstalk to be intelligible, eq. (3), becomes

$$V - L^{ij} > I_0 + N.$$

Now define*

$$R^{ij} \triangleq V - L^{ij} - I_0 - N. \tag{6}$$

Equation (3) is now

$$R^{ij} > 0. \tag{7}$$

* The case of gaussian random variables has been treated previously in detail, as for example in Ref. 1.

2.3 The probability of crosstalk

With that background, the probability of crosstalk over the path L^{ij} can be calculated as the product of the probability of coincident activity over that path times the probability of intelligible crosstalk when active,

$$P_C^{ij} = P_A^{ij} \cdot P_{C|A}^{ij}, \tag{8}$$

where

$$P_{C|A}^{ij} = \Pr\,\{R^{ij} > 0\}. \tag{9}$$

This can be evaluated by expressing the probability density function of R^{ij}, denoted by f_R^{ij}, in terms of the p.d.f.'s of V, L^{ij}, I_0, and N, denoted by f_V, f_L^{ij}, f_{I_0}, and f_N, respectively. By virtue of eq. (6), f_R^{ij} is the convolution

$$f_R^{ij} = f_V * f_{-L}^{ij} * f_{-I_0} * f_{-N}. \tag{10}$$

Restating eqs. (8) and (9) in terms of f_R^{ij},

$$P_C^{ij} = P_A^{ij} \int_0^\infty f_R^{ij}(R)dR. \tag{11}$$

There are $(N - 1)$ possible disturbers when the listener is on pair j. The probability that disturbers on pairs $i \neq j$ exceed the listener's intelligibility threshold is, therefore,

$$P_C^j = 1 - \prod_{i \neq j} (1 - P_C^{ij}).$$

The rare occurrences of babble due to two or more sufficiently loud disturbers are included. When P_C^{ij} are small, this reduces to

$$P_C^j = \sum_{i \neq j} P_C^{ij}.$$

Considering all N possible pairs used by l, the total probability of intelligible crosstalk to subscriber l is, again for small probabilities,

$$\begin{aligned} P_C &= \sum_{j=1}^{N} \sum_{i \neq j} P_C^{ij} \\ &= \sum_{j=1}^{N} \sum_{i \neq j} P_A^{ij} \int_0^\infty f_R^{ij}(R)dR \\ &= \sum_{j=1}^{N} \sum_{i \neq j} \frac{P_A}{N} \int_0^\infty f_R^{ij}(R)dR. \end{aligned} \tag{12}$$

As a result of the linearity of the convolution and integration operations, this double sum of integrals can be reduced to a very simple form. Returning to (6), we can define an intermediate random variable

$$S \triangleq V - I_0 - N, \qquad f_S = f_V * f_{-I_0} * f_{-N}.$$

Thus,

$$R^{ij} \triangleq S - L^{ij}, \qquad f_R^{ij} = f_S * f_{-L}^{ij}.$$

Now (12) can be written as

$$P_C = (N-1)P_A \int_0^\infty f_S * \frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i \neq j} f_{-L}^{ij}(R)dR.$$

The function

$$\frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i \neq j} f_{-L}^{ij}(\cdot)$$

is simply the result of averaging the $N(N-1)$ individual p.d.f.'s f_{-L}^{ij} over all possible channel combinations $i \neq j$. Denote this averaged p.d.f. by f_{-L}:

$$f_L \triangleq \frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i \neq j} f_L^{ij}.$$

Then

$$P_C = (N-1)P_A \int_0^\infty f_S * f_{-L}(R)dR.$$

For

$$R \triangleq V - I_0 - N - L, \qquad f_R = f_V * f_{-I_0} * f_{-N} * f_{-L}, \tag{13}$$

$$P_C = (N-1)P_A \int_0^\infty f_R(R)dR = (N-1)P_A[1 - F_R(0)], \tag{14}$$

where F_R is the probability distribution function of R.

The multidisturber case is thus greatly simplified to a result parallel to the case of a single disturbing path, eq. (11), but with f_R appropriately redefined by means of the average of the coupling loss p.d.f.'s. If all f_L^{ij} are identical, then of course,

$$f_L = f_L^{ij}$$

and eq. (14) differs from eq. (11) only by the factor $(N-1)$. For most cables, however, the f_L^{ij} may not be identical for all pair combinations (i, j), because there may be a strong dependence on distance, shielding due to intervening pairs, and twist-length ratios. Noting that it would be a formidable task to specify the individual f_L^{ij}, we proceed to show how $f_L(L)$ and the result (14) are easily available from a single overall cumulative distribution function.

Since no entirely satisfactory analytic derivation of the f_L^{ij} has yet been devised, the most fruitful approach is to use direct measurements of the coupling losses in cables (more specifically identified as cable units, because within-unit crosstalk predominates). The measurements form a symmetric, off-diagonal matrix for each cable (or cable unit if within-unit coupling is of primary interest), as shown in Fig. 3. By merging these results over all cable units measured, a cumulative distribution function F_L^{ij} in each (i, j) cell can be constructed, from which

$$f_L^{ij}(L) = \frac{d}{dL}[F_L^{ij}(L)]$$

and

$$f_L(L) = \frac{d}{dL}[F_L(L)],$$

where F_L is the overall cumulative distribution function of all crosstalk measurements,

$$F_L(L) \triangleq \frac{1}{N(N-1)} \sum_{j=1}^{N} \sum_{i \neq j} F_L^{ij}(L).$$

Therefore, even though the distributions differ from cell to cell, it is not necessary to maintain the identities (i, j) of the measurements, a point which is most clearly evident when the distribution functions are expressed in terms of step-functions $u(\cdot)$ at the measured values. For K cable units measured,

$$F_L^{ij}(L) = \frac{1}{K} \sum_{k=1}^{K} u(L - L_k^{ij})$$

$$\begin{aligned} F_L(L) &= \frac{1}{N(N-1)K} \sum_{j=1}^{N} \sum_{i \neq j} \sum_{k=1}^{K} u(L - L_k^{ij}) \\ &= \frac{1}{M} \sum_{m=1}^{M} u(L - L_m), \end{aligned}$$

where $M = N(N-1)K$ and the single summation extends over all measurements of all pair combinations.

Section 4.2 points out that F_L may have arisen from two distinct populations, adjacent within-unit (WU) pairs and nonadjacent WU pairs. In that case, proceeding as above, we can model F_L to show this explicitly:

$$F_L(L) = \frac{A}{M} F_{L,a}(L) + \frac{M-A}{M} F_{L,n}(L), \tag{15}$$

where A is the number of adjacent pair measurements.

To summarize, the result (14) is parallel to the case of a single disturbing path in eq. (11), but with f_R appropriately redefined. Of the four functions constituting f_R, as shown in eq. (13), all but the noise power density f_N can be derived directly from measured data. The

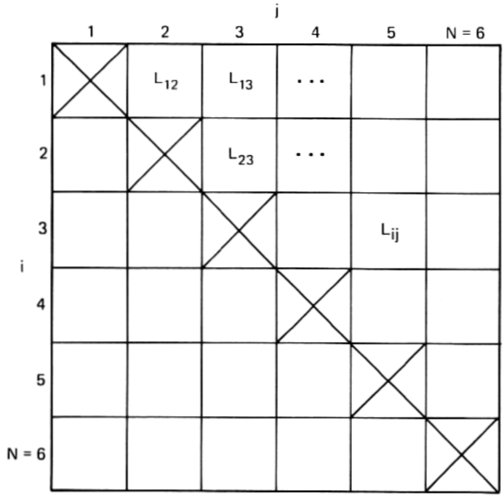

Fig. 3—Symmetric matrix of crosstalk coupling-loss measurements.

calculation of f_N is more complicated and is discussed in the next section.

III. NOISE POWER-SUM CALCULATION

It is necessary to evaluate the probability density of total noise power from stochastically independent noise sources. As summarized in eq. (4), the noise variables N_k, in dB, must be transformed to units of power, e.g., watts,

$$W_k = p(N_k) = 10^{N_k/10} \tag{16a}$$

so that they can be added.

Appendix C describes sums, functions, and general operator sums of independent random variables. To utilize these for the transforming function $p(\cdot)$, we only need to calculate its inverse and derivatives,

$$p^{-1}(W_k) = 10 \log_{10} W_k \tag{16b}$$

$$\frac{d}{dN_k} p(N_k) = \frac{\ln 10}{10} 10^{N_k/10} \tag{16c}$$

$$\frac{d}{dW_k} p^{-1}(W_k) = \frac{10}{\ln 10} \frac{1}{W_k}. \tag{16d}$$

The term N_k and its p.d.f. f_k thus transform as follows:

$$N_k, f_k(N_k) \Leftrightarrow W_k, g_k(W_k) = \frac{10}{\ln 10} \frac{1}{W_k} f_k(10 \log_{10} W_k). \tag{17}$$

Then the total noise power, in units of power, is

$$W = \sum_k W_k$$

and its p.d.f. $g(W)$ is the convolution of the g_k,

$$g = g_1 * \cdots * g_K.$$

To invert W back to dB, we use Appendix C with $u \equiv p^{-1}$ to obtain

$$\begin{aligned} N &= 10 \log_{10} W = 10 \log_{10} \sum_k W_k \\ &= 10 \log_{10} \sum_k 10^{N_k/10} \end{aligned} \tag{18}$$

and

$$\begin{aligned} f_N(N) &= \left\{ \frac{\ln 10}{10} 10^{N/10} \right\} g(10^{N/10}) \\ &= \left\{ \frac{\ln 10}{10} 10^{N/10} \right\} g_1 * \cdots * g_K(10^{N/10}). \end{aligned} \tag{19}$$

This, with the aid of eq. (17), defines the p.d.f. of total noise power.

The power sum convolution process can be summarized by applying it explicitly to noise sources mentioned in Section 2.2. As in Appendix C, the notation can be greatly simplified by introducing the symbol $*_p$ for this "power convolution," analogous to the "power sum" symbol $+_p$ in eq. (4b). Then eqs. (18) and (19) reduce to merely

$$N = N_C +_p N_R +_p N_{TH} \tag{20}$$

$$f_N = f_C *_p f_R *_p f_{TH}. \tag{21}$$

The next step is to describe some data for the random variables discussed: V, L, I_0, N_C, N_R, and N_{TH}.

IV. ESTIMATION OF THE PROBABILITY DISTRIBUTIONS

As mentioned in the preceding sections, the main random variables are talker volume, intelligibility threshold, coupling loss, and noise. Data for them was taken from the best surveys available at this time.

4.1 Talker volume

The 1960 survey of talker volume at central offices as reported by McAdoo[2] was used to model talker volume. Loudness of intraoffice calls varied substantially with the size of the CO and local calling area, the means ranging from −19 VU (volume units) in New York City to −29 VU in Enid, Oklahoma. The figure actually used was −26 VU, as measured in Pascagoula, Mississippi, which has a population of 17,000 with 8000 station sets. This volume level was somewhat below the survey's nationwide average of −24.8 VU, in recognition of the use of VFG (voice-frequency gain) primarily in the smaller communities.

The distribution about the −26 VU mean was modeled as normal with a standard deviation of 7.3 dB, as reported by the survey.

4.2 Coupling loss

Because crosstalk from within-unit (WU) pairs predominates overwhelmingly, the most accurate approach was to consider the distribution of WU coupling loss explicitly, rather than as a small tail of a merged distribution for all pairs in a cable. NEXT coupling loss data at voice frequency was available from Bell Laboratories measurements of an electrically long, 22-gauge, 100-pair, even-count PIC cable, comprising eight units of 12 and 13 pairs. Figure 4 shows the smoothed cumulative distribution function of 468 measurements in WU coupling-loss matrices.

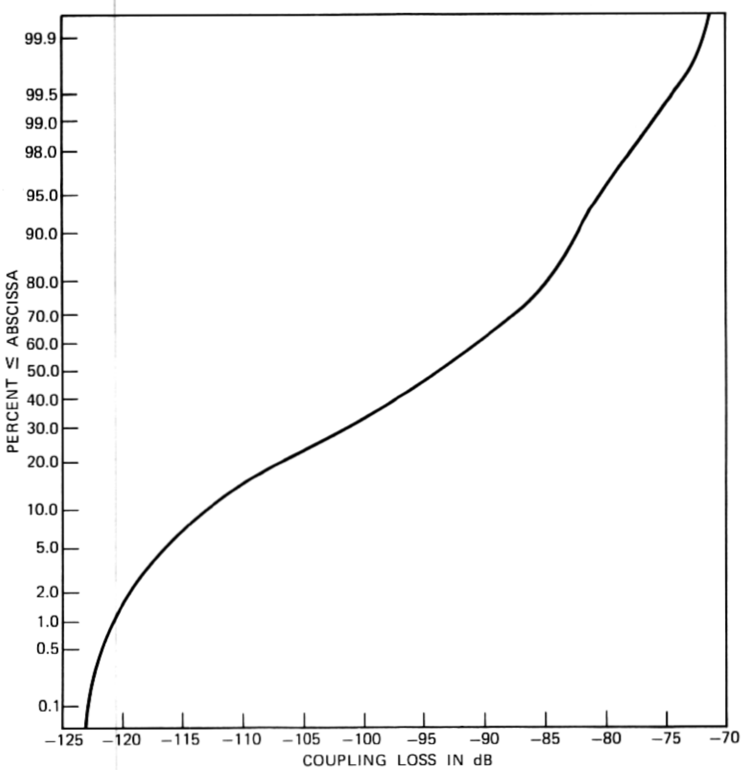

Fig. 4—Distribution of 1000-Hz NEXT coupling loss (dB).

Coupling loss is usually assumed to be log-normally distributed. If this were true of the above data, then it would appear as a straight line in Fig. 4. A possible explanation of why this is realized only approximately is that loss is not distributed uniformly for all pairs, but instead differs for two (or more) distinct populations: adjacent WU pairs and nonadjacent WU pairs. When this hypothesis was tested with some available data of coupling-loss measurements by South Central Bell, a substantial and consistent difference between adjacent-pair and nonadjacent-pair measurements was indeed found.

Therefore, the distribution function of Fig. 4 might be interpreted as a merging of a distribution for adjacent pairs and a distribution for nonadjacent pairs, as in eq. (15). The total distribution function was divided in proportion to the number of adjacent and nonadjacent measurements so that a truncated normal distribution function could be fitted to each part. The approximation gave results that agreed reasonably well with those calculated from the distribution of Fig. 4 exactly (see also Section V); in particular, the truncated normal fit to the low-loss tail, attributed to adjacent pairs, accounted for most of the crosstalk probability.

4.3 Intelligibility threshold

It was mentioned in Section 2.2 that the dependence of intelligibility threshold on total noise level is linear, as developed in Refs. 1 and 3. In this representation, eq. (5), I_0 was modeled as normally distributed in dB with a mean of -95 VU and a standard deviation of 2.5 dB, based on the subjective tests by Sen.[3] These values were determined in experiments in which coupling loss was flat with frequency. The threshold has been found to be slightly lower if coupling loss decreases with frequency; in case of a 6-dB-per-octave slope, sensitivity would be 2 dB greater for a male voice, or 1 dB greater for a female voice. However, in view of other conservative assumptions, this possible effect was neglected.

4.4 Noise sources

Section 3 described the total noise level as a power sum of circuit noise, room noise, and hearing-threshold noise. Their distributions were characterized as described below.

The circuit noise distribution was taken from data of the 1964 Loop Survey described by Gresh.[4] The data for long loops was used because this would be most appropriate to VFG. The distribution is very nearly normal in dB, with a mean of 17.5 dBrnC and a standard deviation of 14.85 dB. However, two adjustments had to be made. First, it is clear from Fig. 2 that the gain would amplify the noise, so that the distribution had to be shifted by this amount. Second, a considerable fraction of the loops exceeded the immediate remedial action limit of 30 dBrnC. Although the excessive noise would very effectively mask crosstalk, it was deemed better to assume that either this plant deficiency has been already cleaned up (for example, by the introduction of ringer isolators), or else it will be done in the near future. Therefore, the distribution used was normal in dB (hence, log-normal in power, as described in Section III and Appendix D), truncated at 30 dBrnC, with a mean of (17.5 + gain) dBrnC, and a standard deviation of 14.85 dB.

For the noise fit parameter N_{TH}, the value was 12.3 dBrnC;* for room noise, a normal distribution with a mean of 45 dBt (dB relative to 20 μPa) and a standard deviation of 7 dB was used.[1] The conversion from dBt to equivalent dBrnC at the set terminals has been

$$45 \text{ dBt} \Rightarrow 11.5 \text{ dBrnC}.$$

Some tests indicate this to be a conservative estimate for mean room noise. However, it is not known whether its inherently different character may make it less masking than circuit noise, so that the conservatively low value of 11.5 dBrnC may be quite appropriate until further experiments are conducted.

V. COMPUTATIONAL ASPECTS

Conceptually, three main steps are involved. First, the "power convolution" of eq. (21) computes the noise power p.d.f. Second, this result is used in the (ordinary) convolution of eq. (13) to compute the p.d.f. of the intelligibility random variable. Finally, the crosstalk probability is easily obtained as the probability that this variable exceeds zero, eq. (14).

Except for very simple p.d.f.'s, such as the normal distributions of V and I_0, these calculations are very difficult to calculate by hand, yet very easy by computer. Consequently, a computer program implementing the calculations of Appendix C was used. It is thereby possible to calculate the p.d.f. of algebraic expressions of independent random variables and their differentiable, invertible transforms; the "power" transform is seen to be a simple special case in Section III. The convolutions involved were performed very conveniently by using a fast Fourier transform routine,[5] a method which is valid to the extent that the p.d.f.'s are well represented by the calculated discrete Fourier transforms.

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* No variance was modeled for this parameter; however, as can be seen from eq. (5), the parameter I_0 does introduce variance due to subjective perceptual differences into the basic equation for the intelligibility threshold.

An advantage of the numerical, rather than analytic, approach is that the p.d.f.'s and transforms could be specified by any method expressible as a function, including the following:

(*i*) Analytic form
(*ii*) Table with analytic interpolation
(*iii*) Distribution of measurement values.

Thus, it is possible to carry out an accurate calculation even when the data is distributed arbitrarily in a way that does not suggest a precise analytic representation. It should be noted also that well-behaved analytic functions can exhibit near-pathological behavior when transformed, as for example in Appendix D, in which case, particular attention to overall numerical accuracy is advisable.

In short, crosstalk probabilities were easily calculated with the data of Section IV used in eqs. (21), (13), and (14), as a special case of Appendix C.

VI. DISCUSSION OF RESULTS

The crosstalk probability calculation involving the three main steps described in Section V was carried out for each value of VFG of interest: 0, 4, 6, and 9 dB. Figures 5 through 8 illustrate the components and results of convolutions in the 4-dB and 9-bB cases. Figures 5 and 6 will serve as an example for the discussion.

The circuit noise p.d.f. with peak at 17.5 dBrnC + 4 dB, the room noise p.d.f. with peak at 11.5 dBrnC, and the convolved total noise p.d.f. including the hearing threshold of 12.3 dBrnC are shown in Fig. 5 for the range 0 to 30 dBrnC. It can be seen that this result is substantially truncated not only near 30 dBrnC, due to truncating the circuit noise there, but also near 12.3 dBrnC, due to the hearing threshold. The latter "truncation," an evident result of the noise power sum, is significant in that it eliminates the possibility of extreme low-noise conditions that would otherwise contribute substantially to the crosstalk probability, or perhaps even dominate it.

Figure 6 shows the convolution of the density of total noise N with the volume above intelligibility $V - I_0$ and with crosstalk coupling loss L. The resulting density is that of the intelligibility random variable R, where $R > 0$ signifies the event that intelligible crosstalk is heard. The probability of this event, as in eq. (14), is shown in Table I for various cases. Results for the three coupling paths identified in Fig. 2 are listed. The NEXT $i \rightarrow l$ case is the one analyzed so far; the probabilities of the FEXT paths are comparatively small, as mentioned in Section 2.1 and analyzed in Appendix A.

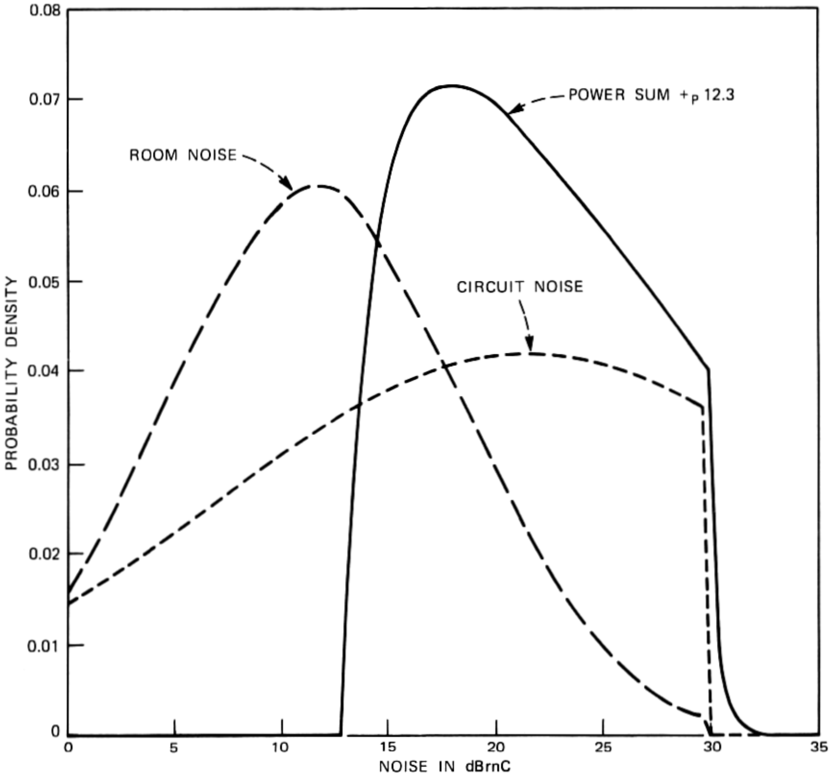

Fig. 5—Power sum of noise sources (gain = 4 dB).

These results show how rapidly the probability rises above the insignificant trace at 0 dB once gain is increased, a feature clearly depicted in Fig. 9. Although a firm objective for crosstalk probability in loop plant has not been established, this should be certainly less than the 1-percent objective for trunk calls because the condition would be dedicated to a particular customer. A probability of 0.1 percent has been used as a criterion.[1] Here we note that this is reached near 4 dB, and is greatly exceeded at 9 dB.

Table I — Probability of intelligible crosstalk

VFG (dB)	NEXT $i \rightarrow l$ (%)	FEXT $m \rightarrow l$ (%)	FEXT $i \rightarrow n$ (%)
0	<0.01	<0.01	<0.01
4	0.14	0.02	0.03
6	0.46	0.04	0.05
9	2.09	0.09	0.18

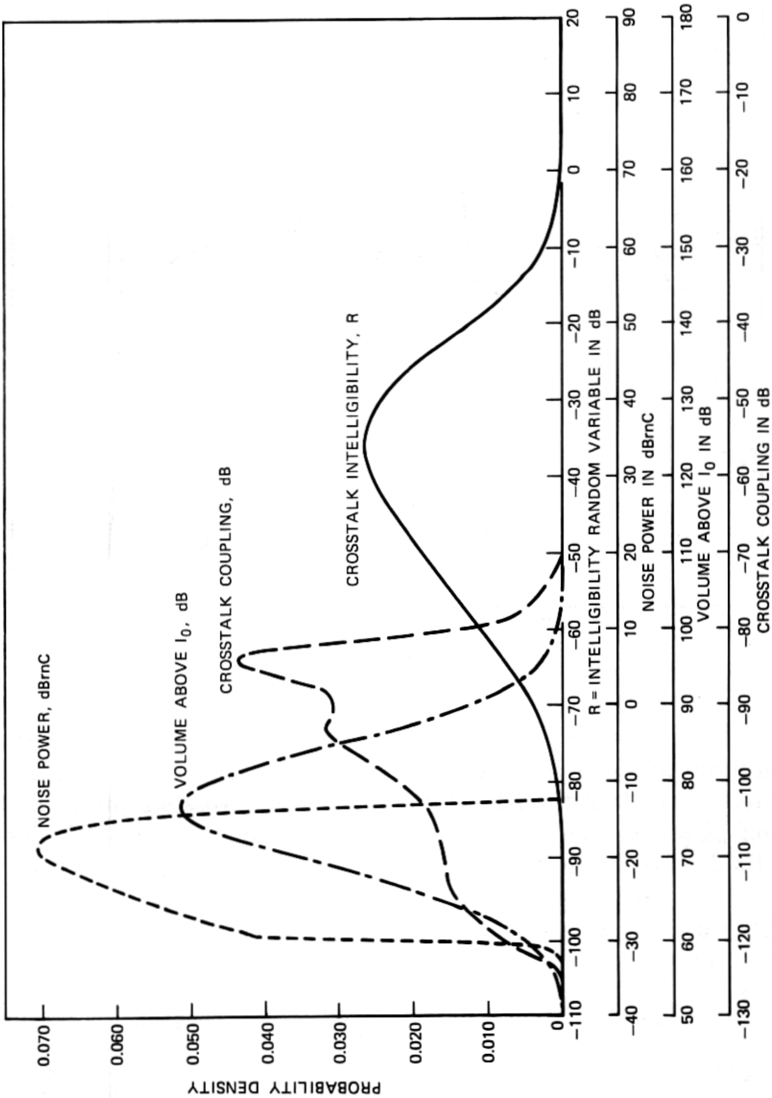

Fig. 6—Crosstalk intelligibility (gain = 4 dB).

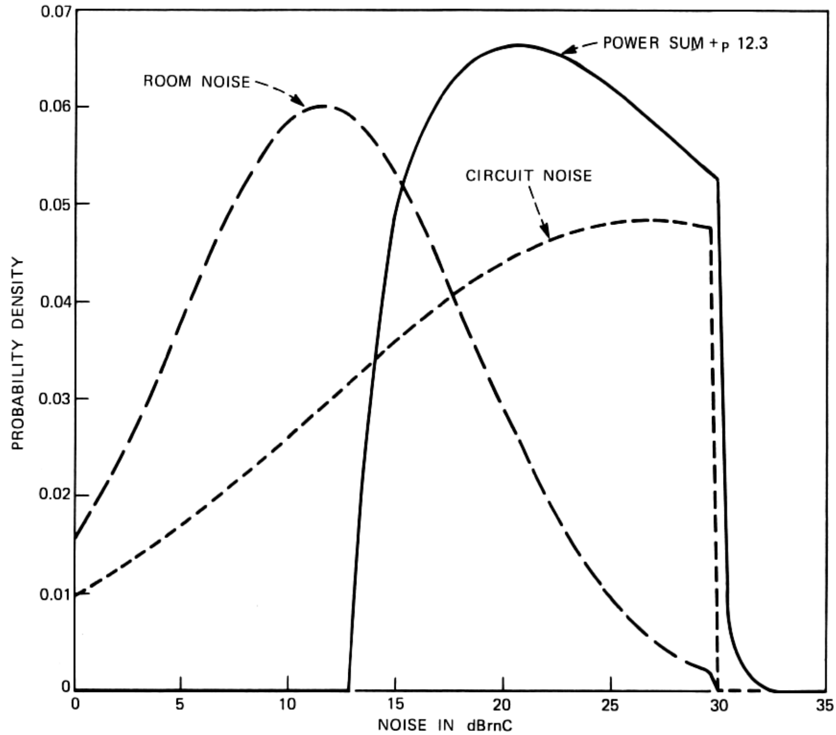

Fig. 7—Power sum of noise sources (gain = 9 dB).

The importance of extreme accuracy in calculation is now apparent. A calculation of the R distribution, which is "only" 99.9-percent accurate, could possibly rate the 4-dB case at *twice* its actual crosstalk probability. Figure 6 shows graphically how small the density is above $R = 0$. In the computer calculation, the precision can be increased as necessary by increasing the order of the fast Fourier transform, so the real limitation is the accuracy of the input data. In that, there is indeed room for further improvement, especially regarding the coupling-loss distribution, which is considered representative according to other measurements but was obtained from a single cable. The true masking effect of room noise should also be investigated further. Fortunately, more complete data are anticipated in the future, and it will be a trivial matter to recalculate the probabilities with this method. However, it is unlikely that the larger probabilities, such as for 9 dB, will turn out sufficiently smaller to be near 0.1 percent.

Four points must be emphasized. First, the crosstalk calculations were made with respect to the connections in Fig. 2—reasonable but

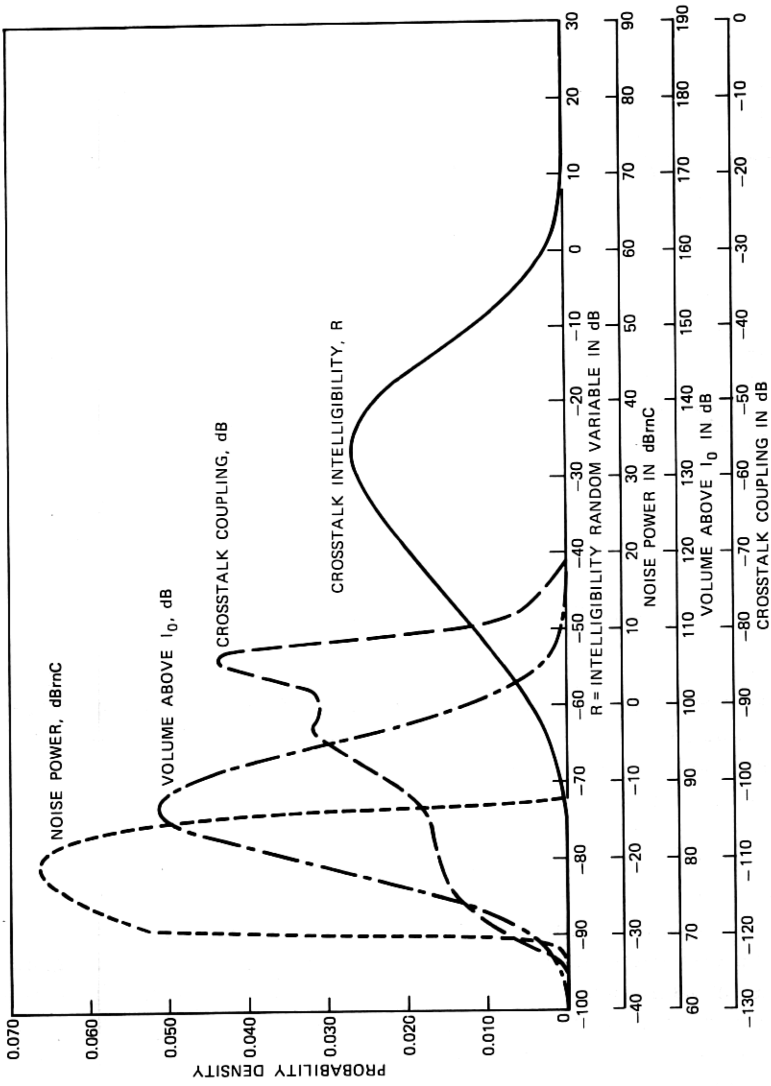

Fig. 8—Crosstalk intelligibility (gain = 9 dB).

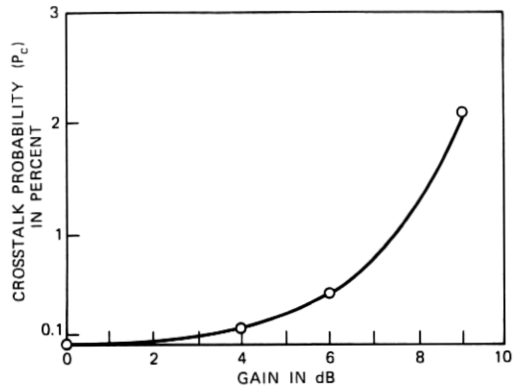

Fig. 9—Total crosstalk probability as a function of gain.

still arbitrary conditions; different conditions would yield different results. Second, the 0.1-percent criterion is intended to be applicable to plant that is fully dedicated to a particular customer. In Fig. 2, not all of customer l's intraoffice calls would be to a community of interest served by VFG. Therefore, provided that the proportion of cables at a CO served by 6-dB gain is less than 20 percent of the cables without gain, the crosstalk probability experienced by the listener will be less than 0.1 percent. Since only 10 percent of the Bell System "long loops" (longer than 30 kft) exceed 60 kft, that degree of saturation is not considered likely; in this case, 6-dB gain is satisfactory. Third, the high noise level of long loops in the 1964 survey (truncated at 30 dBrnC) was used in the calculations. It is likely that extreme noise cases have been and will continue to be mitigated, with the unfortunate side effect of ultimately increasing crosstalk intelligibility. Finally, even under the high circuit-noise level used, the crosstalk probability for 9-dB gain considerably exceeds even the trunk-call objective.

VII. CONCLUSION

The subscriber loop VFG limitation due to probability of crosstalk was evaluated from available data on the random variables involved, including activity coincidence of disturber volume level, coupling-path loss, listener acuity, and various noise sources. Probability distributions which were nongaussian were represented in detail, and the necessary functions, sums, and functions of sums of random variables were calculated accordingly. The results, based on the best data available today, show that gain of 6 dB or less is acceptable and 9 dB is considerably excessive; the desirability of further, more extensive, data is thereby indicated. The method of calculation is independent of specific assumed forms of the distributions of random variables and, hence, anticipated to be convenient not only for similar crosstalk evaluations requiring high accuracy, but also for other non-gaussian probability problems.

VIII. ACKNOWLEDGMENTS

The aid of J. T. Peoples was invaluable in resolving some vexing aspects of the computation method. The study also benefitted from the enlightening comments of many other colleagues at Bell Laboratories, most substantially D. H. Morgen, K. I. Park, and T. C. Spang. The author is grateful to all.

APPENDIX A

FEXT Calculation

Consider the FEXT coupling path shown in Fig. 2, from disturber i to listener n. The calculation is perhaps somewhat subtler than that for NEXT in the main study, because it is necessary to find the distribution of intelligible crosstalk probabilities that might be assigned to the listener, rather than a single value independent of assignment. For a fixed L^{ij}, eq. (11) may be written in terms of $S = V - I_0 - N$,

$$\begin{aligned} P_C^{in} &= P_A^{in} \int_0^\infty f_S(R + L^{in})dR \\ &= P_A \int_{L^{in}}^\infty f_S(S)dS \\ &= P_A[1 - F_S(L^{in})]. \end{aligned} \tag{22}$$

When subscriber n is assigned his pair, he is in effect assigned the coupling loss L^{in}, from some distribution F_L^{in}, and a resulting P_C^{in}. Denote by F_C^{in} this distribution of crosstalk probabilities:

$$\begin{aligned} F_C^{in}(C^{in}) &= P\{P_C^{in} \leqq C^{in}\} \\ &= P\left\{L^{in} \geqq F_S^{-1}\left(1 - \frac{C^{in}}{P_A}\right)\right\} \\ &= F_{-L}^{in}\left[-F_S^{-1}\left(1 - \frac{C^{in}}{P_A}\right)\right]. \end{aligned} \tag{23}$$

Moreover, the disturbed subscriber n will be subject to $N - 1$ such coupling paths. If the individual C^{in} are small, then the total crosstalk probability is approximately

$$C^n = \sum_{i \neq n} C^{in} \tag{24a}$$

with density

$$f_C^n = f_C^{1n} * \dots * f_C^{in} * \dots * f_C^{Nn},\ n, \qquad i \neq n \tag{24b}$$

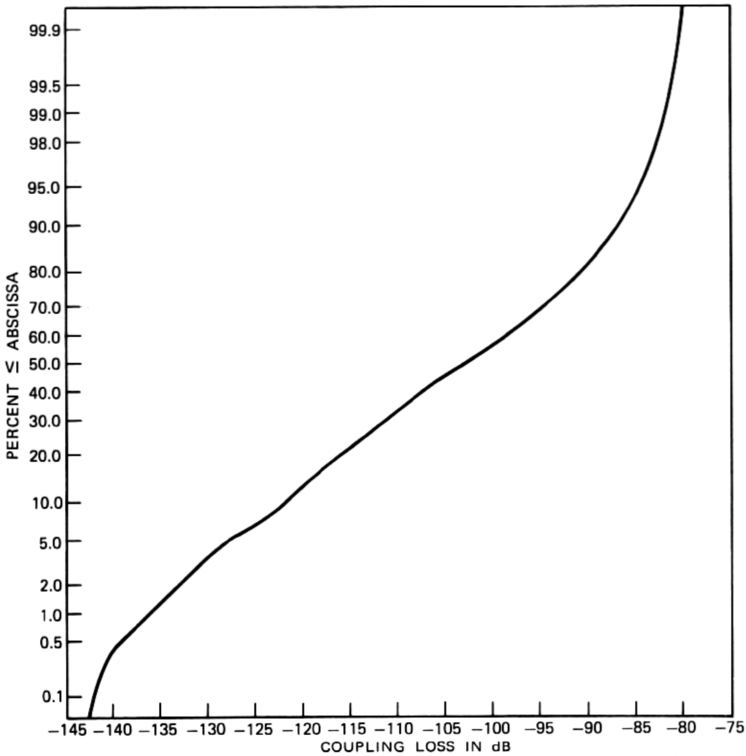

Fig. 10—Distribution of 1000-Hz FEXT coupling loss (dB).

and distribution

$$F_C^n = \int_{-\infty}^{C} f_C^n(C)dC. \tag{24c}$$

Thus, the desired distribution of FEXT probabilities assigned to the subscribers has been determined.

Instead of this calculation, some insight is often gained by evaluating (22) with L^{in} having a fixed value at the worst 1-percent of the coupling-loss distribution. Several sets of past measurements show that the 1-percent worst coupling loss is well estimated to be 84 dB at 6 kft. The loss at other lengths, including gain at the central office, is[6]

$$L^{in}(l) = 84 \text{ dB} - 10 \log_{10}\left(\frac{l}{6 \text{ kft}}\right) + \alpha l - G,$$

where l is the cable length, and α is the attenuation per unit length. The lowest-loss application of 6-dB gain has been for 2000 ohms of

22-gauge cable. This amounts to 11.5 miles, at which length the loss is 9.3 dB.[6] Thus

$$L^{ij}(60.8 \text{ kft}) = 77.24.$$

Using this value, P_c^{in} evaluates as a negligible 0.01 percent. Similar estimates for other gain values are shown below.

We now return to the exact calculation, eqs. (22) through (24). Within-unit FEXT coupling-loss data similar to the NEXT data of Section 4.2 is shown in Fig. 10. However, now the individual distributions F_L^{ij} are required, rather than their merged cumulative distribution as before. Therefore, the calculation is possible only at the price of assuming that the L^{in} are independent and identically distributed according to Fig. 10. The resulting distribution of FEXT probabilities is shown in Fig. 11 for 6-dB gain. It can be seen that the probability of assigning a 0.1 percent or greater FEXT probability to a subscriber is less 0.1 percent.

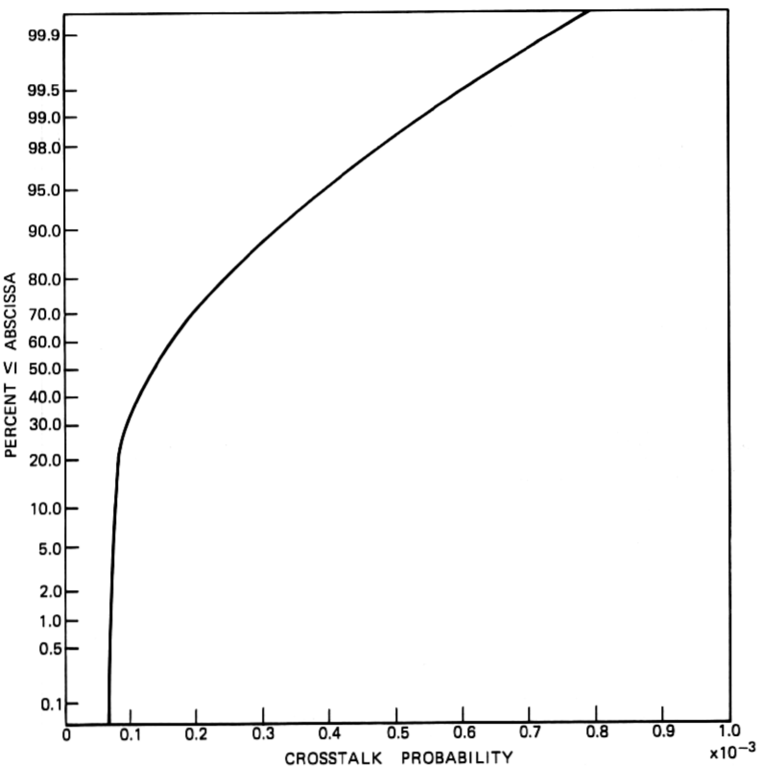

Fig. 11—Distribution of far-end crosstalk probabilities.

Table II — FEXT probability

VFG (dB)	Distributed Loss (%)	1% Worst Loss (%)
0	<0.01	<0.01
4	0.03	<0.01
6	0.05	0.01
9	0.18	0.03

Similar calculations were performed for the other values of gain. Because it is somewhat unwieldy to compare distributions, the 1-percent worst assignment resulting from the distributed coupling loss is specified in Table II. The estimation based on 1-percent worst coupling loss is also shown; both indicate negligible probabilities.

The FEXT coupling path from disturber m to listener l also yielded small crosstalk probabilities. These results are summarized in Section VI.

APPENDIX B

Probability of Coincident Activity of Two Channels

Channels a and b will be assumed to have calls initiated and terminated independently, with the simple assumption that the probabilities of these events during small time increments are proportional to the length of these increments. The quiet intervals and the call holding times will be shown to be exponentially distributed, as in Ref. 7 for a Poisson counting process.

To derive the distribution of quiet intervals, let

$$P_0(x_1, x_2) = \Pr\{\text{No call is initiated in } [x_1, x_2) \mid x_1 \text{ is in a quiet interval}\}.$$

Assume that, as in a Poisson process, the probability of initiation during a time interval Δx is proportional to an initiation-rate parameter $1/\lambda$ except for higher-order terms $0(\Delta x)$,

$$1 - P_0(x, x + \Delta x) = \frac{1}{\lambda}\Delta x + 0(\Delta x) \qquad \forall x \geqq 0,\ \Delta x \geqq 0.$$

The probability of no initiation in $[0, x + \Delta x)$ is then

$$\begin{aligned} P_0(0, x + \Delta x) &= P_0(0, x)P_0(x, x + \Delta x) \\ &= P_0(0, x)\left[1 - \frac{1}{\lambda}\Delta x - 0(\Delta x)\right] \end{aligned}$$

$$\frac{P_0(0, x + \Delta x) - P_0(0, x)}{\Delta x} = -\frac{1}{\lambda} P_0(0, x) - P_0(0, x)\frac{0(\Delta x)}{\Delta x}.$$

In the limit as $\Delta x \rightarrow 0$,

$$\frac{dP_0(0, x)}{dx} = -\frac{1}{\lambda} P_0(0, x),$$

which has the solution

$$P_0(0, x) = e^{-x/\lambda} \qquad x \geqq 0.$$

The probability of initiation in $[x, x + \Delta x)$, with no initiation in $[0, x)$, is then

$$\begin{aligned} P_1(x, x + \Delta x) &= P_0(0, x)[1 - P_0(x, x + \Delta)] \\ &= P_0(0, x) \left[\frac{1}{\lambda} \Delta x + 0(\Delta x) \right] \cdot \end{aligned}$$

This is the probability that the quiet interval after time 0 has a length between x and $x + \Delta x$. Denoting the probability distribution function of these quiet interval lengths by F,

$$\frac{\Delta F(x)}{\Delta x} = \frac{P_1(x, x + \Delta x)}{\Delta x} = P_0(0, x) \frac{1}{\lambda} + \frac{0(\Delta x)}{\Delta x} P_0(0, x).$$

The probability density function of these quiet interval lengths is the limit as $\Delta x \rightarrow 0$,

$$f_x(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta F(x)}{\Delta x} = \frac{1}{\lambda} P_0(0, x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad x \geqq 0. \tag{25}$$

Similarly, given that a channel is active at time 0, the probability density function of a call termination at time y is determined from the termination-rate parameter $1/\mu$,

$$f_y(y) = \begin{cases} \frac{1}{\mu} e^{-y/\mu}, & y \geqq 0 \\ 0, & y < 0 \end{cases}. \tag{26}$$

The mean quiet interval is

$$\int_{-\infty}^{\infty} x f_x(x) dx = \int_0^{\infty} \frac{x}{\lambda} e^{-x/\lambda} dx = \lambda.$$

Similarly, the mean active interval is μ.

The probability to be derived is that of activity on channel b during some portion of a single call on channel a. This is most easily done by first calculating the probability of avoiding such coincidence.

First, the probability that channel a initiates the call during one of b's quiet intervals is

$$\frac{\lambda}{\mu + \lambda}. \tag{27}$$

Next, let y be the time interval from this initiation to the end of a's call, and x be the time interval from this initiation to the end of b's quiet interval. Then activity coincidence is avoided if

$$x \geqq y. \tag{28}$$

Define

$$z = x - y,$$

so that (28) is the event

$$z \geqq 0. \tag{29}$$

The p.d.f. of z is

$$f_z(z) = f_x * f_{-y}(z) = \int_{-\infty}^{\infty} f_x(x) f_{-y}(z - x) dx.$$

From (25) and (26),

$$\begin{aligned} f_z(z) &= \int_z^{\infty} \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{(z-x)/\mu} dx, \qquad \text{for} \quad z \geqq 0 \\ &= \frac{1}{\lambda\mu} e^{z/\mu} \int_z^{\infty} e^{-x[(1/\lambda)+(1/\mu)]} dx \\ &= \frac{1}{\lambda + \mu} e^{-z/\lambda}. \end{aligned} \tag{30}$$

From (27) and (29), the probability of avoiding coincident activity is

$$\begin{aligned} 1 - P_A^{ab} &= \left[\frac{\lambda}{\mu + \lambda}\right] \int_0^{\infty} f_z(z) dz \\ 1 - P_A^{ab} &= \left[\frac{\lambda}{\mu + \lambda}\right] \int_0^{\infty} \frac{1}{\lambda + \mu} e^{-z/\lambda} dz \\ &= \left[\frac{\lambda}{\mu + \lambda}\right]^2. \end{aligned}$$

Thus,

$$P_A^{ab} = 1 - \left[\frac{\lambda}{\mu + \lambda}\right]^2.$$

It should be noted that this is a conditional probability describing the event of coincidence given that channel a is active. In other words, this is the probability that b is active during any portion or portions of precisely one call by a. It is necessary to be aware of just how "probability of coincident activity" is defined to avoid ambiguous or misleading results.

APPENDIX C

Functions and Operator Sums of Independent Random Variables

Let X be a random variable with p.d.f. $f(X)$. Let $u(\cdot)$ be an invertible function such that u and its inverse are differentiable. Then the p.d.f. of

$$Y = u(X)$$

denoted by $g(Y)$ is easily found as follows:

$$\Pr\{y \leqq Y\} = \Pr\{u(x) \leqq Y\} = \Pr\{x \leqq u^{-1}(Y)\}$$
$$= \int_{-\infty}^{u^{-1}(Y)} f(X)dX$$
$$g(Y) = \frac{d}{dY}\int_{-\infty}^{u^{-1}(Y)} f(X)dX = \left[\frac{d}{dY}\,u^{-1}(Y)\right] f[u^{-1}(Y)]. \tag{31}$$

Now consider the problem of finding the p.d.f. of a random variable that is the inverse function of the sum of a function of several random variables,

$$X = u^{-1}[u(X_1) + \cdots + u(X_n)] = u^{-1}[\sum_i u(X_i)], \tag{32}$$

with the p.d.f. of X_i denoted by f_i. The first step is to find the p.d.f. of

$$Y_i = u(X_i), \tag{33}$$

which is

$$g_i(Y_i) = \left[\frac{d}{du}\,u^{-1}(Y_i)\right] f[u^{-1}(Y_i)]. \tag{34}$$

Then the p.d.f. of

$$Y = \sum_i Y_i \tag{35}$$

is the convolution

$$g(Y) = g_1 * \cdots * g_n(Y), \tag{36}$$

and the p.d.f. of $X = u^{-1}(Y)$ is, by applying eq. (31) with $u(\cdot)$ replaced by $u^{-1}(\cdot)$,

$$f(X) = \left[\frac{d}{dX}\,u(X)\right] g[u(X)]. \tag{37}$$

The calculations (32) through (37) are much more conveniently expressed by defining an operator $+_u$ analogous to $+$, to deal with the function $u(\cdot)$:

$$(X_a) +_u (X_b) \triangleq u^{-1}[u(X_a) + u(X_b)].$$

Similarly, the extended convolution $*_u$ is defined by eqs. (34), (36), and (37). Then all of (32) through (37) can be expressed as

$$X = X_i +_u \cdots +_u X_n$$
$$f(X) = f_1 *_u \cdots *_u f_n.$$

Section III illustrates the convenience of this notation in the case of power sums. Another important application is the multiplication of random variables,

$$X = X_a \cdot X_b = e^{\log X_a + \log X_b} = X_a +_m X_b,$$

where the function $m(\cdot)$ is defined as

$$m(X) = \log X,$$

which is invertible and differentiable as required. Thus,

$$f(X) = f_a *_m f_b.$$

Since both addition and multiplication are covered, it is possible to calculate in this manner the p.d.f. of algebraic expressions of independent random variables and their differentiable, invertible transforms.

APPENDIX D

Some Pathological Aspects of Power-Summing Random Variables with Large Variances

When the variances of gaussian random variables in a power sum are *small* compared to the means, it is easy to see that the power function will not substantially distort their distributions. With $p(\cdot)$ defined as in Section III,

$$Y = p(X) = 10^{X/10}, \qquad X = q(Y) = p^{-1}(Y) = 10 \log_{10} Y.$$

The term Y can be normalized with respect to its mean μ and standard deviation σ if the variance is finite:

$$\mathbf{Y} \triangleq \frac{Y - \mu}{\sigma}$$

$$X = q(Y) = 10 \log_{10} (\mu + \sigma \mathbf{Y}) = \mathbf{q}(\mathbf{Y}).$$

If $N(X)$ is the normal c.d.f. of X, and $G(\mathbf{Y})$ the c.d.f. of $\mathbf{Y}$, by Appendix C they are related as follows,

$$G(\mathbf{Y}) = N[\mathbf{q}(\mathbf{Y})].$$

Expanding around $\mathbf{Y} = 0$,

$$\mathbf{q}(\mathbf{Y}) = \mathbf{q}(0) + \frac{d\mathbf{q}}{d\mathbf{Y}}(0)\mathbf{Y} \quad \text{with error} < \left|\frac{d^2\mathbf{q}}{d\mathbf{Y}^2}(0)\mathbf{Y}^2\right|$$

$$\mathbf{q}(\mathbf{Y}) = 10 \log_{10} \mu + \frac{10\sigma}{\ln 10\mu}\mathbf{Y} \quad \text{with error} < \frac{10}{\ln 10}\left(\frac{\sigma}{\mu}\right)^2 \mathbf{Y}^2.$$

Thus, for $|\sigma/\mu|$ small $\mathbf{q}$ is approximately linear, and the c.d.f. of $\mathbf{Y}$

may be expected to approach

$$G(\mathbf{Y}) = N\left(10\log_{10}\mu + \frac{10\sigma}{\ln 10\mu}\,\mathbf{Y}\right),$$

which is normal also.

An entirely different situation exists when the variances are *large* compared to the means. As an example, consider the distribution of circuit noise N_C from Section IV, which is normal in dB with p.d.f.

$$F_C(N_C) = \frac{1}{\sqrt{2\pi}\sigma}\exp[-(N_C-\mu)^2/2\sigma^2], \qquad \mu = 17.5, \quad \sigma = 14.85.$$

Applying $p(\cdot)$ to find the density in units of power,

$$\begin{aligned} g_C(W_C) &= \frac{10}{\ln 10}\frac{1}{W_C}f_C(10\log_{10}W_C) \\ &= \frac{10}{\ln 10}\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{W_C}\exp[-\{(10\log_{10}W_C-\mu)^2/2\sigma^2\}]. \end{aligned}$$

To show that this density is extremely skewed, we find its maximum by differentiating,

$$\begin{aligned} \frac{dg_C(W_C)}{dW_C} &= \frac{10}{\ln 10}\frac{1}{\sqrt{2\pi}\sigma}\left[\frac{1}{W_C}\left(-\frac{1}{\sigma^2}\right)(10\log_{10}W_C-\mu)\right. \\ &\quad \left.\cdot\left(\frac{10}{\ln 10}\frac{1}{W_C}\right)-\frac{1}{W_C^2}\right]\exp\left[-\{(10\log_{10}W_C-\mu)^2/2\sigma^2\}\right]. \end{aligned}$$

This is zero at

$$W_{C\,\mathrm{Max}} = 10^{[\mu-\sigma^2(\ln 10/10)]/10} = 10^{[17.5-14.85^2(\ln 10/10)]/10} = 0.0005.$$

Now the *median* value of g_C is at

$$W_{C\,\mathrm{Med}} = p(\mu) = 10^{\mu/10} = 10^{17.5/10} = 87.5.$$

The large difference between the mode and the median indicates the function's skewness.

A graph of g_C up to the truncation point 30 dBrnC would show very little, but the salient features are depicted in Fig. 12, which has the scale broken and expanded at $W_C = 10$ and 0.003 so that some detail is visible. It can be seen that g_C is less than 10 percent of the maximum over almost all of the range 0 to 1000.0, but rises sharply for values near zero.

Because of the difference between the mode and the median of g_C, it can be concluded that neither a normal density at the median nor an impulse at the maximum would be a very good approximation.

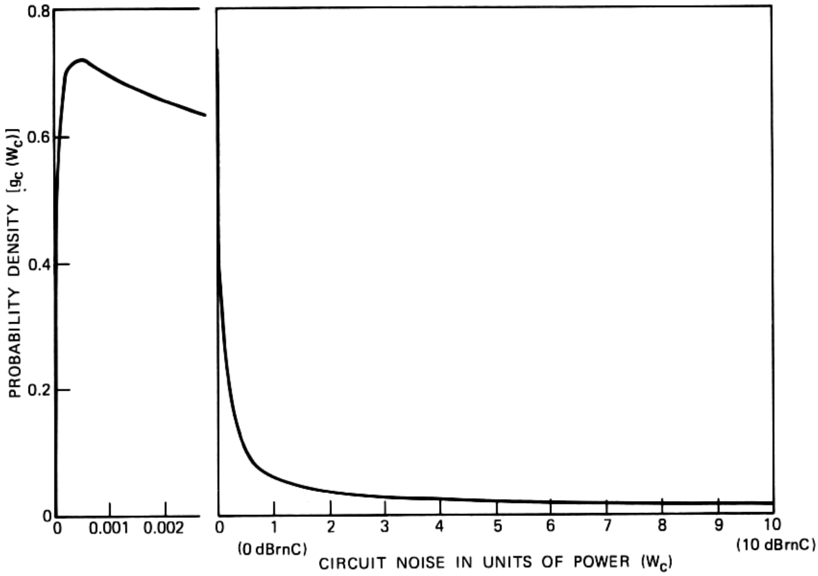

Fig. 12—Circuit noise density on expanded scales.

The former would underestimate the density at the low-noise tail, whereas the latter would overestimate it.

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